MATH 3060 Tutorial 10

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December 1, 2021

In the tutorial, we discussed the following:

Theorem 0.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that for each $x \in \mathbb{R}$, there exists some $n \in \mathbb{N}$ with $f^{(n)}(x) = 0$. Then f is a polynomial.*

The proof basically follows the outline here. You can also check this post for more interesting applications of Baire category theorem.

Throughout the proof, we will freely use the following two facts:

(i) If $a < b < c < d$, and g is a polynomial on both (a, c) and (b, d), then f is a polynomial on (a, d).

(ii) If g is differentiable and $g = 0$ on a subset E of $\mathbb{R}$, then $g' = 0$ on E' (The set of limit points of E).

For the first item, say $f = p_1$ on (a, c) and $f = p_2$ on (b, d) be the two polynomials, then $p_1 = p_2$ on (b, c) and so p_1 and p_2 are the same polynomials. (Two different polynomials can only agree on a finite set)

For the second item, let $x' \in E'$, then we can choose $x_n \in E$ and $x_n \to x$. then $g'(x) = \lim_n \frac{f(x_n)-f(x)}{x_n - x} = \lim_n \frac{0-0}{x_n-x} = 0$.

Proof. We follows the steps in the questions, we introduce the notations:and showing the following steps:

$$X = \{x \in \mathbb{R} : f \text{ is not a polynomial in any open neighbourhood of } x\},$$

$$S_n = \{x \in \mathbb{R} : f^{(n)}(x) = 0\},$$

and show the following

Step 1: Show that X is closed without isolated points.

Step 2: Show that S_n is closed.

Step 3: Suppose $X \neq \emptyset$. Show that there exists a positive integer n and a nonempty open interval (a, b) such that

$$\emptyset \neq (a, b) \cap X \subset S_n.$$

Step 4: Show that f is a polynomial.

We first show that how step 1-3 can implies step 4.
Suppose f is not a polynomial, then X is nonempty by item (i). By Step 3, we can find a nonempty open interval (a,b) such that

$$\emptyset \neq (a,b) \cap X \subset S_n.$$

We claim that $f^{(n)} = 0$ on (a,b). This will imply f is a polynomial of degree $< n$ on (a,b), contradicting to the assumption $(a,b) \cap X \neq \emptyset$.

To show the claim, let $x \in (a,b)$. If $x \in X$, then $x \in S_n$ and so we are done.

Now suppose $x \notin X$. By Step 1, we know that $(a,b) \setminus X$ is open, so we can find a maximal open interval $(a',b') \subset (a,b) \setminus X$ so that $x \in (a',b')$. (The existence of maximal interval follows from question 1 tutorial 10.) We note that either $a' \in X$ or $b' \in X$, because otherwise we must have $a' = a$ and $b' = b$ contradicting to $(a,b) \cap X \neq \emptyset$. Let's assume $a' \in X$
Since $(a',b') \subset X^c$, we know f equals some polynomial of some degree d on (a',b') by item (i). In particular, we have $f^{(d)}$ is a nonzero constant on (a',b'). By continuity, we must also have

$$f^{(d)}(a') \neq 0.$$

We finally make use of item (ii): $f^{(n)} = 0$ on $(a,b) \cap X$, but we know X has no isolate points (i.e. $X' = X$). So item (ii) says $f^{(m)} = 0$ for all $m \geq m$. This gives $d < n$ and hence $f^{(n)}(x) = 0$.

To finish the proof, we must show step 1-3.

Step 1: We first show X^c is open. Let $x \in X^c$, then f is a polynomial in a neighbourhood $(x-\epsilon, x+\epsilon)$, and so $(x-\epsilon, x+\epsilon) \subset X^c$. We next show that X has no isolated points. In fact, if $x \in X$ is an isolated points, then we can find a neighbourhood $(x-\epsilon, x+\epsilon)$ so that $(x-\epsilon, x+\epsilon) \cap X = \{x\}$. But then f is a polynomial on $(x-\epsilon, x)$ and on $(x, x+\epsilon)$, so we can find positive integers n_1, n_2, n_3 so that $f^{(n_1)} = 0$ on $(x-\epsilon, x)$, $f^{(n_3)} = 0$ on $(x, x+\epsilon)$ and $f^{(n_2)}(x) = 0$. So if we take $n = \max\{n_1, n_2, n_3\}$, we have $f^{(n)} = 0$ on $(x-\epsilon, x+\epsilon)$ which contradicts to $x \in X$.

Step 2: Since $f^{(n)}$ is continuous and $\{0\}$ is closed, so $S_n = (f^{(n)})^{-1}(\{0\})$ is closed.

Step 3: Step 1 tells us that X is closed subset of the complete metric space $\mathbb{R}$, so X is a complete metric space. On the other hand,

$$X = \cup_{n=1}^{\infty} (X \cap S_n),$$

so by Baire category theorem, some $X \cap S_n$ has nonempty interior. In other words, it contains some open subsets of X, i.e.

$$\emptyset \neq (a,b) \cap X \subset X \cap S_n \subset S_n.$$

for some $a, b \in \mathbb{R}$.

$\square$